CS395T: Continuous Algorithms
Homework V

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Due date: April 24, 2025, start of class (3:30 PM).

Please list all collaborators on the first page of your solutions. Unless we have discussed and I have specified otherwise, homework is not accepted if it is not turned in by hand at the start of class, or turned in electronically on Canvas by then. Send me an email to discuss any exceptions.

1 Problem 1

Let $D_\alpha(P\|Q) := \frac{1}{\alpha-1}\log(\int_\Omega P(\omega)^\alpha Q(\omega)^{1-\alpha}\mathrm{d}\omega)$ denote the Rényi divergence of order $\alpha \geq 1$ between two probability distributions P, Q supported on the same sample space Ω.

(i) Prove that if $\alpha \leq \beta$, then $D_\alpha(P\|Q) \leq D_\beta(P\|Q)$. You may use that for fixed P, Q, $\log(\int_\Omega P(\omega)^\alpha Q(\omega)^{1-\alpha}\mathrm{d}\omega)$ is convex as a function of α.

(ii) Prove that $\lim_{\alpha\to 1^+} D_\alpha(P\|Q) = D_{\mathrm{KL}}(P\|Q)$,[1] assuming $D_\alpha(P\|Q) < \infty$ for some $\alpha > 1$.

(iii) Let $f : \Omega \to \Omega'$ be arbitrary, and let P^f denote the density of $f(\omega)$ for $\omega \sim P$, and similarly define Q^f. Prove that for all $\alpha \geq 1$,

$$\int_{\Omega'} \left(\frac{P^f(\omega')}{Q^f(\omega')}\right)^\alpha Q^f(\omega')\mathrm{d}\omega' \leq \int_\Omega \left(\frac{P(\omega)}{Q(\omega)}\right)^\alpha Q(\omega)\mathrm{d}\omega$$

and conclude that $D_\alpha(f(P)\|f(Q)) \leq D_\alpha(P\|Q)$.

2 Problem 2

The *underdamped Langevin dynamics* (ULD) is the drift-diffusion process on $\{(\mathbf{x}_t, \mathbf{v}_t)\}_{t\geq 0} \subset \mathbb{R}^d \times \mathbb{R}^d$ satisfying the following stochastic differential equation, for a parameter $\gamma > 0$:[2]

$$\mathrm{d}\begin{pmatrix}\mathbf{x}_t \\ \mathbf{v}_t\end{pmatrix} = \begin{pmatrix}\mathbf{v}_t \\ -\mathbf{v}_t - \gamma\nabla V(\mathbf{x}_t)\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\mathbf{0}_d & \mathbf{0}_d \\ \mathbf{0}_d & \sqrt{2\gamma}\mathbf{I}_d\end{pmatrix}\mathrm{d}\boldsymbol{B}_t,$$

where $\{\boldsymbol{B}_t\}_{t\geq 0}$ is Brownian motion in $\mathbb{R}^{2d}$, and $V : \mathbb{R}^d \to \mathbb{R}$ satisfies $\int \exp(-V(\mathbf{x}))\mathrm{d}\mathbf{x} < \infty$.

(i) Let $\mathcal{L}$ be the generator of the ULD. Give a formula for $\mathcal{L}f$, for a smooth function $f : \mathbb{R}^{2d} \to \mathbb{R}$.

(ii) Let $\mathcal{L}^*$ be the adjoint of $\mathcal{L}$. Give a formula for $\mathcal{L}^*\pi$, for a probability density $\pi \in \mathcal{P}(\mathbb{R}^{2d})$.

(iii) Verify that

$$\pi^\star(\mathbf{x}, \mathbf{v}) \propto \exp\left(-V(\mathbf{x}) - \frac{1}{2\gamma}\|\mathbf{v}\|_2^2\right)$$

is a stationary distribution for the ULD.

[1] We use $\lim_{\cdot\to\cdot^+}$ to denote a one-sided limit from the right.

[2] Note the similarity to the accelerated gradient flow ODE (Proposition 2, Part V). Coincidence?

3 Problem 3

(i) Let $\pi \in \mathcal{P}(\mathbb{R}^d)$, and let $f : \mathbb{R}^d \to \mathbb{R}$ satisfy $\int f(\mathbf{x})^2 \pi(\mathbf{x}) \mathrm{d}\mathbf{x} < \infty$. Prove that

$$\mathrm{Var}_\pi [f] = \inf_{m \in \mathbb{R}} \mathbb{E}_\pi \left[(f(\mathbf{x}) - m)^2 \right].$$

(ii) Let $\mu, \pi \in \mathcal{P}(\mathbb{R}^d)$ have $c \le \frac{\mu(\mathbf{x})}{\pi(\mathbf{x})} \le C$ for all $\mathbf{x} \in \mathbb{R}^d$, and let π satisfy a Poincaré inequality with constant C_{PI}. Prove that μ satisfies a Poincaré inequality with constant $\frac{C}{c} \cdot C_{\mathrm{PI}}$.

4 Problem 4

Let $\pi = \mathcal{N}(\boldsymbol{\mu}, \mathbf{A})$, $\pi' = \mathcal{N}(\boldsymbol{\nu}, \mathbf{B})$ be multivariate normal densities on $\mathbb{R}^d$, with respective means and full-rank covariance matrices $(\boldsymbol{\mu}, \mathbf{A}) \in \mathbb{R}^d \times \mathbb{S}^{d \times d}_{\succ \mathbf{0}}$ and $(\boldsymbol{\nu}, \mathbf{B}) \in \mathbb{R}^d \times \mathbb{S}^{d \times d}_{\succ \mathbf{0}}$.

(i) Brenier's theorem (see Proposition 4, Part XIII) states there is a unique optimal transport map $\nabla \varphi$ sending $\mathbf{x} \sim \pi$ to $\mathbf{y} = \nabla \varphi(\mathbf{x}) \sim \pi'$, for convex $\varphi : \mathbb{R}^d \to \mathbb{R}$, such that $(\mathbf{x}, \nabla \varphi(\mathbf{x}))$ for $\mathbf{x} \sim \pi$ is the coupling realizing $W_2^2(\pi, \pi')$. What is φ in the above setting?[3]

(ii) Compute $W_2^2(\pi, \pi')$ in the above setting.

5 Problem 5

Please fill out this form: `https://forms.gle/iKttijhGYkgngZsR9`.

[3] The identity $\mathbf{A}^{-\frac{1}{2}} (\mathbf{A}^{\frac{1}{2}} \mathbf{B} \mathbf{A}^{\frac{1}{2}})^{\frac{1}{2}} \mathbf{A}^{-\frac{1}{2}} = \mathbf{A}^{-1} (\mathbf{A}\mathbf{B})^{\frac{1}{2}}$ may be helpful.